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A CLASS OF BOUNDARY-VALUE PROBLEMS IN THE DYNAMIC THEORY OF ELASTICITY†

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A class of boundary-value problems of the dynamic theory of elasticity, which has not been studied to any great extent, is investigated. In these problems all the components of the displacement vector and the stress vector are specified on part of the boundary of the body (nothing is known about the components of these fields on the remaining part of the boundary). A uniqueness theorem is proved. The problem is investigated by reducing the boundary-value problem to a system of Fredholm integral equations of the first kind with smooth kernels. A scheme for the numerical determination of the unknown fields is proposed, based on a combination of the boundary-element method and the Tikhonov regularization method. A number of model examples in establishing the wave fields for an anisotropic body are considered.

As is well known, for a body, bounded by a surface S, in the classical dynamic theory of elasticity, there are three main types of boundary-value problems for which theorems of existence and uniqueness have been proved [1]. The development of a procedure for establishing the wave fields inside elastic bodies based on measured boundary wave fields leads to new formulations of the boundary-value problems in the dynamic theory of elasticity. The most important one is the boundary-value problem in which all the components of the displacement vector and all the components of the stress vector are specified on part of the boundary $S_1 \subset S$. Boundary-value problems of this type arise when investigating inverse boundary-value problems of the theory of elasticity, related to determining the boundary displacement and stress fields on a part $S_2 = S/S_1$ of the elastic body inaccessible for direct observation, and also when establishing the stress and displacement fields inside the body from the specified (measured) displacement field on the load-free part of the boundary [2–5]. Note that problems of uniqueness, correctness and the construction of algorithms for establishing the unknown fields are extremely important aspects when investigating such boundary-value problems.

1. FORMULATION OF THE PROBLEM

We will assume that an elastic body V is bounded by a smooth surface S, $S = S_1 \cup S_2$, where the region V and its boundary S satisfy the following conditions:

(a) V is a bounded region, which is a combination of regions each of which is a star with respect to a certain sphere;

(b) the surface S can be split into a finite number of parts Σ_k, each of which can be projected uniquely onto some coordinate plane using a continuously differentiable mapping, defined in a closed region $\overline{g_k}$.

The boundary-value problem of establishing the oscillations of an isotropic body V with frequency ω is described by the following system of equations [1]

$$Lu = c_{ijkl}u_{k,lj} + \rho\omega^2 u_i = 0, \quad i = 1,2,3 \tag{1.1}$$

and the boundary conditions on S_1

$$u_i |_{S_1} = u_{i0}, \quad t_i = c_{ijkl}u_{k,l}n_j |_{S_1} = p_i, \quad i = 1,2,3 \tag{1.2}$$

where c_{ijkl} are the components of the elastic-constant tensor, which satisfy the usual requirements of being positive definite and symmetrical, and n_j are the components of the unique vector of the outward normal to the surface S. On the part of the boundary S_2 neither the type of boundary conditions nor the distribution of the boundary values of the displacement and stress fields are known. It is required

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to determine the boundary values of the displacement and stress vector on the boundary S_2. Knowing the values of the boundary displacement and stress fields over the whole surface S, it is easy to calculate their values inside V using Somigliana's formulae [1].

Note that the formulation of boundary-value problem (1.1), (1.2) is not the traditional one in mathematical physics for elliptic-type equations and requires an investigation.

2. REDUCTION TO A CAUCHY PROBLEM

We will show that boundary-value problem (1.1), (1.2) can easily be reduced to a Cauchy problem for an elliptic operator of the theory of elasticity. Suppose $x_0 \in S_1$ is an internal point of S_1. We will introduce a local system of coordinates with centre at $x_0 - (s_1, s_2, n)$, where s_1 and s_2 are directed along the lines of principal curvature, and n is in the direction of the outward normal to S_1. Then, from boundary conditions (1.2), taking into account the fact that

$$\frac{\partial}{\partial x_j} = m_{1j}\frac{\partial}{\partial s_1} + m_{2j}\frac{\partial}{\partial s_2} + n_j\frac{\partial}{\partial n}$$

we obtain

$$u_i\,|_{S_1} = u_{i0},\quad c_{ijkl}n_l n_j\frac{\partial u_k}{\partial n}\,|_{S_1} = p_{i0} \tag{2.1}$$

$$p_{i0} = p_i - c_{ijkl}m_{1l}n_j\frac{\partial u_{k0}}{\partial s_1} - c_{ijkl}m_{2l}n_j\frac{\partial u_{k0}}{\partial s_2},\quad i = 1,2,3$$

In view of the fact that the elastic energy is positive definite, system (2.1) is uniquely solvable with respect to $\partial u_k/\partial n$, and relations (2.1) reduce to the form

$$u_i\,|_{S_1} = u_{i0},\quad \frac{\partial u_i}{\partial n}\,|_{S_1} = \mathrm{v}_{i0},\quad \mathrm{v}_{i0} = (c_{ijkl}n_l n_j)^{-1}p_{i0} \tag{2.2}$$

Hence, the initial boundary-value problem (1.1), (1.2) reduces to Cauchy problem (2.2) for system (1.1). We know that this problem is ill-posed for one elliptic-type equation and is not locally solvable [6]. Problem (1.1), (1.2) possesses the same property, but, using discussions similar to those used to prove Holmgren's theorem [6], we can establish the uniqueness of the solution of problem (1.1), (1.2).

3. UNIQUENESS OF THEOREM

Theorem. Suppose a solution $u_i \in C^2(V)$ of problem (1.1), (1.2) exists. Then, it is unique in this class.

To prove this it is sufficient to how that $u_i = 0$ inside the region V when $u_{i0} = p_i = 0$ by virtue of the linearity of the problem. We will introduce an arbitrary piecewise-smooth surface $S_0 = S_1^- \cup S_1^+$ which cuts from V a part V_0, where $S_1^- \subset S_1$ and S_1^+ inside V. In addition, we will introduce a set of vector-functions $v_i^{(m)}$ which are solutions of the the following uniform boundary-value problems.

$$L\mathrm{v}^{(m)} = 0,\quad x \in V_0$$

$$\mathrm{v}_i^{(m)}\,|_{S_1^+} = 0,\quad c_{ijkl}n_j^+\frac{\partial \mathrm{v}_k^{(m)}}{\partial x_j}\,|_{S_1^+} = Q_i\delta_{im}$$

where Q_i is an arbitrary polynomial on S_1^+.

Using Gauss' theorem and the boundary conditions for u_i and $v_i^{(m)}$ we calculate the difference

$$(Lu,\mathrm{v}^{(m)}) - (u, L\mathrm{v}^{(m)}) = \int_{V_0}\left[\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\mathrm{v}_i^{(m)} - \frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial \mathrm{v}_k^{(m)}}{\partial x_l}\right)u_i\right]dV =$$

$$= \int_{S_0}\left[\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)\mathrm{v}_i^{(m)}n_j^+ - \left(c_{ijkl}\frac{\partial \mathrm{v}_k^{(m)}}{\partial x_l}\right)u_i n_j^+\right]dS = \int_{S_1^+} Q_i(x)\delta_{im}u_i\,dS_x = 0$$

Hence since $Q_i(x)$ is arbitrary it follows that $u_m(x)|_{S_1^+} = 0$, and since S_1^+ is arbitrary we have $u_m(x) \equiv 0$ in V_0, which proves the theorem.

Remark 1. The uniqueness theorem remains true for a non-uniform medium with $c_{ijk}(x) \in C^2(V)$.

4. CONSTRUCTION OF THE SOLUTION

Since problem (1.1), (1.2) is not a classical boundary-value problem, we cannot use the present well-developed finite-element method to solve it. The first stage in the procedure for establishing the displacement and stress fields inside the region V is to determine the boundary fields on the part of the boundary S_2. One of the most effective methods of finding the relation between the known and unknown boundary values of the displacements and stresses is the method of boundary integral equations and the boundary-element method based on it [7]. In view of the lack of an explicit representation of the fundamental solutions for an isotropic medium, the classical approach, based on potential theory, is ineffective, but it is possible to formulate an operator relation between the boundary values of the displacement and stress vectors, based on the use of boundary integral equations of the first kind with smooth exponential-type kernels [8]. As in [8] we can construct a system of operator equations, on the basis of which we can formulate a system of boundary integral equations of the first kind with smooth kernels for boundary-value problem (1.1), (1.2) in the following form

$$\int_{S_2} K_{mj}^{(1)}(x,\alpha)t_m(x)dS_x - \int_{S_2} K_{mj}^{(2)}(x,\alpha)u_m(x)dS_x = F_{j0}(\alpha), \quad j=1,2,3 \tag{4.1}$$
$$\alpha = (\alpha_1, \alpha_2) \in C^2$$

where

$$\begin{aligned} &K_{mj}^{(1)}(x,\alpha) = P_m(\alpha_1,\alpha_2,\lambda_j)\phi_j(x,\alpha) \\ &K_{mj}^{(2)}(x,\alpha) = c_{ksml}P_k(\alpha_1,\alpha_2,\lambda_j)n_l\frac{\partial\phi_j(x,\alpha)}{\partial x_s} \\ &\phi_j(x,\alpha) = \exp(i(\alpha_1 x_1 + \alpha_2 x_2 + \lambda_j(\alpha_1,\alpha_2)x_3)) \\ &F_{j0}(\alpha) = \int_{S_1} K_{mj}^{(2)}(x,\alpha)u_{m0}(x)dS_x - \int_{S_1} K_{mj}^{(1)}(x,\alpha)p_m(x)dS_x \end{aligned} \tag{4.2}$$

Here $\lambda_j(\alpha_1, \alpha_2) = \alpha_{3j}(\alpha_1, \alpha_2)$ $(j = 1, 2, 3)$ are the roots of Christoffel's equation

$$\det A = 0, \quad A = \{A_{ik}\}, \quad A_{ik} = c_{ijkl}\alpha_j\alpha_l - \rho\omega^2\delta_{ik}$$

which satisfy the condition $\operatorname{Im}\lambda_j(\alpha_1, \alpha_2) > 0$ as $\alpha_1^2 + \alpha_2^2 \to \infty$, while $P_m(\alpha_1, \alpha_2, \lambda_j)$ are fourth-order polynomials, which are the cofactors of the elements of the first row of matrix A.

The concrete form of kernels (4.2) of the integral operators in (4.1) were derived previously in [9–11] for isotropic and various anisotropic materials. Note that the kernels are piecewise-smooth functions, unlike the classical boundary integral equations, the kernels of which have singularities.

System (4.1) is a system of Fredholm integral equations of the first kind with smooth kernels with respect to the unknown boundary fields, which is equivalent to the initial boundary-value problem. These equation are a consequence of the reciprocity theorem in the theory of elasticity for the true fields and non-uniform plane waves in an anisotropic medium. System (4.1) generates a completely continuous operator, and the procedure for inverting it is an ill-posed problem and requires regularization in some form [12]. In this paper we find the unknown functions by a combination of the boundary-element method and the Tikhonov regularization method, as is done for classical boundary-value problems of the dynamic theory of elasticity in the isotropic case [9] and the anisotropic case [10,11].

Remark 2. As is well known, for ill-posed problem (1.1), (1.2), generally speaking, there is no continuous dependence on the data of the problem u_{i0} and p_i but, if we are seeking a solution in the class of uniformly bounded functions in the region V, there will be a continuous dependence [13].

5. A MODEL EXAMPLE

We will consider, as a model example, the simplest problem of the type being investigated, namely, the problem of antiplane deformation of an orthotropic body. The boundary-value problem of establishing the displacement field in this case has the form

$$c_{66}u_{,11} + c_{44}u_{,33} + \rho\omega^2 u = 0 \tag{5.1}$$

$$x_3 = a_3, \quad u = f^{(1)}(x_1), \quad u_{,3} = f^{(2)}(x_1), \quad |x_1| \leq a_1$$

We will establish the field in the region $S = [-a_1, a_1] \times [a_3, a_3]$. The solution of problem (5.1) is easily constructed by the method of separation of variables and can be represented in the form

$$u(x_1, x_3) = u_0(x_1, x_3) + u_1(x_1, x_3) + u_2(x_1, x_3) \tag{5.2}$$

where

$$u_0(x_1, x_3) = f_0^{(1)}(x_1)\cos k(x_3 - a_3) + f_0^{(2)}(x_1)\frac{\sin k(x_3 - a_3)}{k}$$

$$u_1(x_1, x_3) = \sum_{n=1}^{M} W(x_1, x_3, n), \quad u_2(x_1, x_3) = \sum_{n=M+1}^{\infty} W(x_1, x_3, n)$$

$$W(x_1, x_3, n) = \left(f_n'^{(1)} \cos\mu_n(x_3 - a_3) + f_n'^{(2)} \frac{\sin\mu_n(x_3 - a_3)}{\mu_n} \right) \sin\lambda_n x_1$$

$$f_0^{(i)}(x_1) = \frac{f^{(i)}(a_1) + f^{(i)}(-a_1)}{2a_1} x_1 + \frac{f^{(i)}(a_1) - f^{(i)}(-a_1)}{2a_1}$$

$$f_n'^{(i)} = \frac{1}{a_1} \int_{-a_1}^{a_1} (f^{(i)}(x_1) - f_0^{(i)}(x_1)) \sin\lambda_n x_1 dx_1; \quad i = 1, 2$$

$$\lambda_n = \frac{\pi n}{a_1}, \quad \mu_n = \sqrt{\frac{(k^2 - \lambda_n^2)}{\nu}}, \quad n = 1, 2, \ldots, \quad k^2 = \frac{\rho\omega^2}{c_{44}}, \quad \nu = \frac{c_{66}}{c_{44}}$$

Note the following features of solution (5.2).

1°. A solution of the form (5.2) exists for any values of the wave number k, unlike classical boundary-value problems for which there is a denumerable spectrum of resonance values of k.

2°. The term $u_0(x_1, x_3)$ is the rod solution, for each fixed $x_1 \in (-a_1, a_1)$ it is solution of the Cauchy problem for a semi-infinite rod $u_{0,11} = 0$.

3°. In the representation for $u_1(x_1, x_3)$ the upper limit of M is chosen from the condition

$$\lambda_M < k < \lambda_{M+1}$$

and $u_1(x_1, x_3)$ is the sum of certain natural forms of oscillation for a rectangle.

4°. The expression for $u_1(x_1, x_3)$ is, generally speaking, a diverging series, which is due to the general result that there is no global solution.

At the same time, by requiring the solution to be bounded, we obtain that it is necessary to confine ourselves to a finite number of terms for $u_2(x_1, x_3)$ in the representation

$$u(x_1, x_3) = u_0(x_1, x_3) + u_1(x_1, x_3)$$

6. NUMERICAL REALIZATION OF THE SYSTEM OF BOUNDARY INTEGRAL EQUATIONS

As examples, which illustrate the effectiveness of the system of boundary integral equations (4.1) for the problem of establishing the boundary fields (1.1), (1.2), we will consider the plane deformation of an orthotropic elastic body. We will assume that the orthotropic axes coincide with the coordinate axes, $u_1 = u_1(x_1, x_3)$, $u_3 = u_3(x_1, x_3)$, $u_2 = 0$. Austenite steels possess orthotropic properties [11], as well as many composite materials within the framework of the concept of effective moduli.

Consider the boundary-value problem of the oscillations of an orthotropic elastic medium, occupying a region S, bounded by a piecewise-smooth curve $L = L_1 \cup L_2$. We will assume that the components of the displacement vector and the stress vector

$$u_i \,|_{L_1} = u_{i0}, \quad \sigma_{ij} n_j \,|_{L_1} = p_i, \quad i = 1,3 \tag{6.1}$$

are specified on the contour L_1, while the components $u_i \;|L_2$ and $t_i = \sigma_{ij} n_j \;|t_2$ $(i = 1, 3)$ are unknown on L_2. In this case, in the same way as the approach described, we can formulate a system of operator relations, which connect the known and unknown values of the boundary fields.

In dimensionless form system (4.1) becomes

$$\begin{aligned} &P_1(\beta_1, \pm\beta_{3s}(\beta_1)) V_1(k\beta_1, \pm k\beta_{3s}(\beta_1)) + \\ &+ P_3(\beta_1, \pm\beta_{3s}(\beta_1)) V_3(k\beta_1, \pm k\beta_{3s}(\beta_1)) = 0, \quad s = 1,2 \end{aligned} \tag{6.2}$$

where

$$P_1(\beta_1, \beta_3) = \gamma_5 \beta_1^2 + \beta_3^2 - 1, \quad P_3(\beta_1, \beta_3) = -(\gamma_5 + \gamma_7)\beta_1\beta_3$$

$$V_1(k\beta_1, k\beta_3) = \int_L \{\sigma_{11} n_1 + \sigma_{13} n_3 - ik[(\beta_1 n_1 \gamma_1 +$$

$$+ \beta_3 n_3 \gamma_5) u_1 + (\beta_1 n_3 \gamma_7 + \beta_3 n_1 \gamma_5) u_3]\} e^{ik(\beta, x)} \, dL_x$$

$$V_3(k\beta_1, k\beta_3) = \int_L \{\sigma_{31} n_1 + \sigma_{33} n_3 - ik[(\beta_1 n_3 \gamma_5 +$$

$$+ \beta_3 n_1 \gamma_7) u_1 + (\beta_1 n_1 \gamma_5 + \beta_3 n_3) u_3]\} e^{ik(\beta, x)} \, dL_x$$

$$k = \omega\sqrt{\rho / c_{33}}, \quad \gamma_1 = c_{11} / c_{33}, \quad \gamma_5 = c_{44} / c_{33}, \quad \gamma_7 = c_{13} / c_{33}$$

$$\beta_{3s}(\beta_1) = i[A_1(\beta_1) - i(-1)^s (A_2(\beta_1))^{1/2}]^{1/2}, \quad s = 1,2$$

$$A_1(\beta_1) = (2\gamma_5)^{-1}[(\gamma_1 - 2\gamma_5\gamma_7 - \gamma_7^2)\beta_1^2 - (1 + \gamma_5)]$$

$$A_2(\beta_1) = -(A_1(\beta_1))^2 + (\gamma_5)^{-1}(1 - \gamma_1\beta_1^2)(1 - \gamma_5\beta_1^2)$$

The kernels of the integral operators in (6.2) depend on the roots of the characteristic polynomial of the operator of the orthotropic theory of elasticity $\beta_{3s}(\beta_1)$ $(s = 1, 2)$.

In Fig. 1 we show curves of $\beta_{3s}(\beta_1)$ for austenite steel with the following material constants [11]

$$\rho = 0.812 \times 10^4 \text{ kg/m}^3$$
$$c_{11} = 0.2627, \; c_{13} = 0.145, \; c_{33} = 0.216, \; c_{44} = 0.129 \times 10^{12} \text{ N/m}^2$$

Note that for the chosen material for small β_1 the roots $\beta_{3s}(\beta_1)$ are real; as β_1 increases one of the roots becomes pure imaginary on a certain part; further, beginning at a certain β_{1*}, the roots $\beta_{3s}(\beta_1)$ $(s = 1, 2)$ are complex conjugate.

As examples, which illustrate the reconstruction of the boundary fields, we will consider two plane problems for square and elliptical regions of orthotropic material with the above elasticity constants.

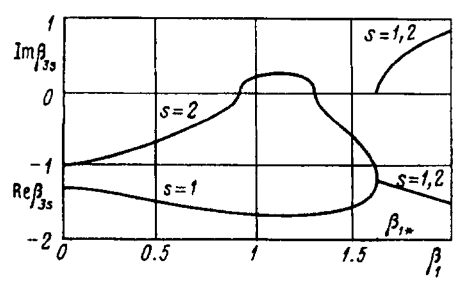

Fig. 1

1. *The problem for a square*

$$S = [0, a] \times [0, a]$$

On

$$L_1 = \{\{x_1 = 0, x_3 \in [0, a]\} \cup \{x_3 = a, x_1 \in [0, a]\} \cup \{x_1 = a, x_3 \in [0, a]\}\}$$

we know the functions u_{i0} and p_i (i=1, 3) from (6.1), corresponding to the displacement field

$$\begin{aligned} &u_1(x_1, x_3) = \mathrm{Re}\{-P_3 Z(x_1, x_3)\}, \quad u_3(x_1, x_3) = \mathrm{Re}\{P_1 Z(x_1, x_3)\} \\ &Z(x_1, x_3) = \exp[ik(\beta_1 x_1 + \beta_3 x_3)], \quad \beta_3 = \beta_{31}(\beta_1), \quad P_i = P_i(\beta_1, \beta_3), \quad i = 1, 3 \end{aligned} \tag{6.3}$$

and the stress field $\sigma_{ij}(x_1, x_3)$, $(i, j = 1, 3)$ calculated using the generalized Hooke's law for an orthotropic material.

The quantities which are to be established on

$$L_2 = \{x_1 \in [0, a]; \; x_3 = 0\}$$

are $u_i(x_1, 0)$, $\sigma_{i3}(x_1, 0)$ (i = 1, 3). In Fig. 2(a) for

$$a = 1, \quad ka = 1.5, \quad k\beta_1 = 2, \quad k\beta_3 = 2.4823$$

we show graphs of the functions $u_i(x_1, 0)$ and $u_3(x_1, 0)$ on the boundary L_2, the continuous curves correspond to the exact solution (6.3) and the dashed curves correspond to values established numerically. In Fig. 2(b) we show similar curves for $\sigma_{13}(x_1, 0)$ and $\sigma_{33}(x_1, 0)$. The calculations were carried out by splitting the boundary L_2 into 20 elements. The results of the calculations show that the reconstructed fields in the problem considered are quite accurate over the range of variation of the parameter *ka* from 0.1 to 5 (the error is less than 10%).

2. *The problem for an ellipse*

$$S = \{x_1, x_3 \mid \frac{(x_1 - R)^2}{a^2} + \frac{(x_3 - R)^2}{b^2} \leq 1\}$$

When a = 0.5 and b = 0.3, we will take as the test reconstructed fields the components defined by relations (6.3) in the case when

$$ka = 0.9, \quad k\beta_1 = 2, \quad k\beta_3 = -1.4964 + 1.0172i$$

We investigated the effectiveness of the proposed method of reconstructing the elastic fields on the boundary

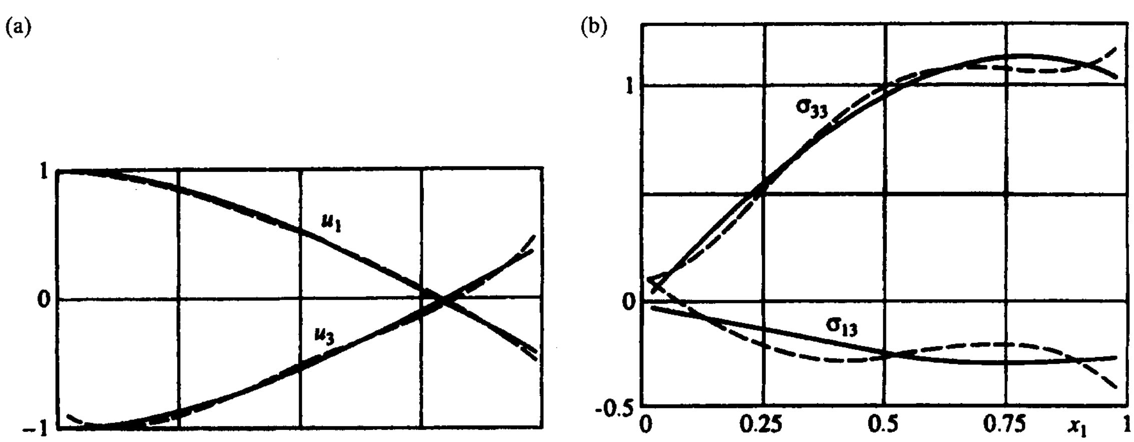

Fig. 2

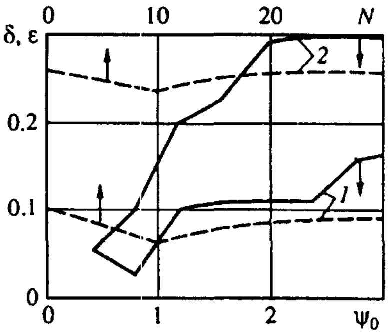

Fig. 3

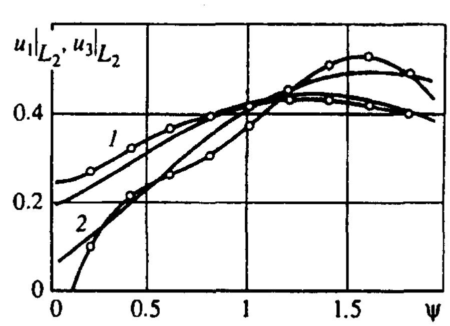

Fig. 4

$$L_2 = \{x_1 = R + a\cos\psi,\ x_3 = R + b\sin\psi,\ \psi \in [0, \psi_0]\}$$

as a function of the number of boundary elements N and as a function of the relative length of the arc of the boundary

$$L_1 = \{x_1 = R + a\cos\psi,\ x_3 = R + b\sin\psi,\ \psi \in [\psi_0, 2\pi]\}$$

accessible for measurements of the characteristics of the elastic fields.

In Fig. 3 the dashed curves represent graphs of the relative errors δ (curve 1) and ε (curve 2) as a function of the number of boundary elements N

$$\delta = \sqrt{\delta_1^2 + \delta_3^2},\quad \delta_i = \frac{\max|u_i - u_i^N|}{\max|u_i|},\quad i = 1,3$$

$$\varepsilon = \sqrt{\varepsilon_1^2 + \varepsilon_2^2},\quad \varepsilon_1 = \frac{\max|\sigma_n - \sigma_n^N|}{\max|\sigma_n|},\quad \varepsilon_2 = \frac{\max|\sigma_\tau - \sigma_\tau^N|}{\max|\sigma_\tau|}$$

when reconstructing the components of the displacement and stress vectors on the boundary L_2 with $\psi_0 = \pi/2$ (u_i, σ_n, σ_τ is the accurate solution, u_i^N, σ_n^N, σ_τ^N are the reconstructed values and the maximum is taken on the boundary L_2). The results of the calculations show that there is an optimum value of the number of elements N_{opt}, which gives the minimum reconstruction error (in this case $N_{opt} = 10$), which corresponds to the results obtained in [5]. Note that when the number of elements N is increased both the errors increase monotonically, which is due to the ill-posed nature of the initial problem (1.1), (1.2).

In addition, we investigated the effectiveness of the proposed method of reconstructing the elastic fields as a function of the angle ψ_0. The unknowns are the components of the displacement and stress vectors on the boundary L_2. In Fig. 3 the continuous curves represent graphs of the relative errors δ (curve 1) and ε (curve 2) when reconstructing the components of the displacement and stress vectors on the boundary L_2 for $\psi_0 \in [\pi/8, \pi]$. In this case we used the concept of constant boundary elements, while the central angle with aperture $\psi = \pi/40$ corresponds to one element.

A number of calculations on reconstructing the elastic fields confirm that they are determined fairly accurately when the length of the part L_1 exceeds the length of the part L_2 by a factor of 3 or more; the reconstruction accuracy falls when there is a relative increase in the length of the part L_2.

As can be seen from Fig. 4, in which for $\psi_0 = 5\pi/8$ we show graphs of the functions $u_1 |_{L_2}$ (curves 1) and $u_3 |_{L_2}$ (curves 2), where the small circles denote the reconstructed values, the greatest error in this numerical method arises at the ends of the line L_2; inside this range the error in reconstructing the field does not exceed 12%. The jumps in the required values at the edges are characteristic of Tikhonov's regularization method in the class of summed functions when solving Fredholm integral equations of the first kind with smooth kernels.

These examples of the reconstruction of the elastic fields confirm that the proposed numerical reconstruction algorithm is quite effective.

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